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The Gamma Function and Stirling's Formula Revisited

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The gamma function $\Gamma(x)$ may be defined in several ways. An elementary one to define $\Gamma(x)$ as the unique solution of the functional equation $f(x + 1) = xf(x)$ has been developed by Artin, who deduced from it many properties of $\Gamma(x)$, e.g. Stirling's formula. The purpose of this article is to show elementary methods of obtaining Stirling's formula, some of which have recently drawn my attention.

§ 1. Functional Equations

Define for $x>0$ the gamma function $\Gamma(x)$ by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \tag{0}$$

As is well known $\Gamma(x)$ plays an important role in mathematical analysis as well as in number theory, and (0) can be defined for complex values of x with real part positive.

One of the most useful properties of $\Gamma(x)$ in applications is Stirling's formula, i.e. in its simplest form,

$$(1) \qquad \Gamma(x) \sim \sqrt{2\pi}\, x^{x-\frac{1}{2}}\, e^{-x}, \quad (x \to \infty).$$

Even the special case of (1) ($\Gamma(n + 1) = n!$),

$$(2) \qquad n! \sim \sqrt{2\pi}\, n^{n+\frac{1}{2}}\, e^{-n}, \quad (n \to \infty)$$

is only desirable in many cases.

It was Bohr and Mollerup [3] who first characterized $\Gamma(x)$ by the (unique) solution of the functional equation

$$(3) \qquad \begin{aligned} f(x + 1) &= xf(x) \\ f(1) &= 1 \end{aligned}$$

with additional conditions on $f(x)$. This idea of introducing $\Gamma(x)$ by (3) was later noticed by Artin, who further developed the theory to deduce many known properties of $\Gamma(x)$ [1][2]. The starting point is the result that if $f(x) > 0$ for $x>0$ is log-convex, i.e. $\log f(x)$ is convex, and satisfies (3), then $f(x) = \Gamma(x)$. Note that (3) does not yield a unique solution if we instead assume $f(x)$ is real analytic. Indeed,

$$f(x) = \Gamma(x) \exp(A(x)),$$

where $A(x)$ is an arbitrary periodic analytic function with period 1 and $A(1) = 1$, always satisfies (3).

The crucial point to obtain Stirling's formula is to prove

(4) $$f(\tfrac{1}{2}) = \sqrt{\pi},$$ (*)

for which Artin gave two different proofs. In Kuczma ([7] Chap.XI) is reproduced the one leading to (4) via "Legendre's relation"

(5) $$2^{x-1} f(\tfrac{x}{2})f(\tfrac{x+1}{2}) = f(\tfrac{1}{2})f(x).$$

The following lemma plays an essential part in the proof.

Lemma. The only C^1 solution of the equation

(6) $$\phi(\tfrac{x}{2}) + \phi(\tfrac{x+1}{2}) = \phi(x), \quad x>0,$$

which is periodic with period 1, is $\phi(x) \equiv 0$.

Artin showed ([2] Theorem 6.2.) that the mere continuity of $\phi(x)$ is not sufficient to ensure the unique solution.
For example,

(7) $$\phi(x) = \sum_{n=1}^{\infty} 2^{-n} \sin(2^n \pi x),$$

which is continuous but not a constant, satisfies (6).

(*) From (0), it is plain that (4) is equivalent to

$$\int_0^{\infty} \exp(-u^2)\, du = \sqrt{\pi}/2.$$

A stronger assertion may be in fact true. For example, in the lemma the hypothesis $\phi \in C^1$ cannot be relaxed to $\phi \in \mathrm{Lip}$, i.e., ϕ of ordinary Lipschitz class (Proposition B below).

It is natural and convenient to consider more generally the functional equation

$$(6') \qquad \phi\left(\frac{x}{2}\right) + \phi\left(\frac{x+1}{2}\right) = \alpha\, \phi(x),$$

where $\phi(x)$ is assumed periodic with period 1 and α is a positive constant.

Then the following propositions about the solution $\phi(x)$ of (6′) are obtained in an elementary way.

A) When $0<\alpha<1$, there exists a non-constant $\phi \in C^1$.

B) When $\alpha = 1$, there exists a non-constant $\phi \in \mathrm{Lip}$.

C) When $1<\alpha<2$, there exists a non-constant continuous ϕ, while $\phi \equiv 0$ is the only C^1 solution.

D) When $\alpha \geqq 2$, there exists a non-constant Riemann integrable ϕ, while $\phi \equiv$constant is the only continuous solution.

It is worth remarking that the proof in Artin [2] is invalid as it stands, if $\alpha<1$. Further, general theorems as presented in [7] seem to be inapplicable if $\alpha<2$, for example.

§ 2. Stirling's formula for n!

In most textbooks (2) is obtained,depending upon some existence theorem of limit ,through Wallis' formula:

$$(8) \qquad \frac{2\cdot 4\cdot 6\cdots\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n-1)} \sim \sqrt{\pi n}\,, \quad (n \to \infty).$$

A somewhat different approach to (2) has recently been proposed by Kurokawa [8], who employed the trapezoidal rule for definite integral as well as (8). The method was refined and sharpened by Hitotumatu [5]. He reproduced the proof of (2) by Kurokawa in the new edition of his textbook([6] Chap. 6), where he comments in the footnote that the main advantage of it over the previously existing proofs is that it leads to (2) straightforward avoiding any consideration on the existence of limit, while keeping its elementary and simple feature. It was after reading the book when another elementary proof, which appears to have been hitherto escaped attention, has come into my notice. I found the proof in Todhunter([11] Chap.XVI). From a historical view-point, the following commentary by Todhunter may be noteworthy: From Wallis's formula we may deduce in an elementary way the formula for the approximate value of 1.2.3....x, when x is very large. Professor De Morgan seems to have first noticed this in his Differential and Integral Calculus, page 293; and the process has been put in a very simple form by Serret: see his Cours de Calcul Differentiel et Integral, Vol. II. page 206.

Although I was unable to see both De Morgan and Serret as cited above in my university library, I found there the German translation of Serrets' instead [10]. The proof given in Todhunter is essentially the same as in Serret, and I will describe the central procedure of it with a slight modification.

Put

(9) $$\phi_n = \frac{n!}{e^{-n}\, n^n \sqrt{2\pi n}} .$$

Then (8) implies

(10) $$\phi_n^2 \;/\; \phi_{2n} \to 1 , \quad (n \to \infty).$$

The thing is to prove

(11) $$\phi_{2n} \;/\; \phi_n \to 1, \quad (n \to \infty)$$

and together with (10) obtain $\phi_n \to 1$.

Since by definition

$$\frac{\phi_n}{\phi_{n+1}} = \frac{1}{e} \left(1 + \frac{1}{n} \right)^{n+\frac{1}{2}} ,$$

it follows that

$$\log\frac{\phi_n}{\phi_{n+1}} = -1 + (n + \frac{1}{2})\log(1 + \frac{1}{n}).$$

Hence the estimate

$$(12) \qquad \log(1+x) = x - \frac{x^2}{2} + O(x^3), \quad (x \to 0+)$$

leads to

$$\log\frac{\phi_n}{\phi_{n+1}} = -1 + (n + \frac{1}{2})\{\frac{1}{n} - \frac{1}{2n^2} + O(\frac{1}{n^3})\} = O(\frac{1}{n^2}).$$

Therefore

$$\log\frac{\phi_n}{\phi_{n+1}} + \log\frac{\phi_{n+1}}{\phi_{n+2}} + \cdots + \log\frac{\phi_{2n-1}}{\phi_{2n}} = O(n\frac{1}{n^2}) = O(\frac{1}{n}),$$

i.e.

$$\log\frac{\phi_n}{\phi_{2n}} = O(\frac{1}{n}),$$

which implies (11), and the proof of (2) is complete.

It is worth mentioning that a slight modification of the above argument will give a much more precise estimate such as

$$(13) \qquad n! = \sqrt{2\pi}\, n^{n+\frac{1}{2}}\, e^{-n+\frac{\theta}{12n}}, \quad (0<\theta<1).$$

Also it will be remarked that a similar proof can be seen in some textbooks (cf. e.g. [4] [9]). Fujiwara ([4] Chap.4) attributes it to Cesàro, Algebraische Analysis, 1904, p.154. However, such a direct calculation as mentioned above is not shown in them.

§ 3. Stirling's formula for $\Gamma(x)$

In this section I will present an elementary way, due essentially to Serret([10] Kap.IV §5.), to prove (1). I have only replaced his argument on uniform convergence by a direct estimation so that the proof might maintain its thorough elementariness as a whole.

Assume Gauss' formula

$$\Gamma(x) = \lim_{n\to\infty} \frac{n^{x-1}\, n!}{x(x+1)\cdots(x+n-1)}, \tag{14}$$

which can be deduced, for example, from (3) and log-convexity of $f(x)$ in an elementary way as shown in [2].

Obviously (14) is equivalent to

$$\Gamma(x) = \frac{n^{x-1}\, n!}{x(x+1)\cdots(x+n-1)} \left(1 + \frac{1}{n}\right)^{x-1} (1 + o(1)),$$

as $n \to \infty$, because $\lim_{n\to\infty} (1 + 1/n)^{x-1} = 1.$

This may be written as

$$\Gamma(x) = \left(\frac{2}{1}\right)^{x-1} \frac{1}{x} \left(\frac{3}{2}\right)^{x-1} \frac{2}{x+1} \cdots \left(\frac{n+1}{n}\right)^{x-1} \frac{n}{x+n-1} (1+o(1)),$$

or equivalently,

$$\log \Gamma(x) = \sum_{k=1}^{n} \left\{ (x-1)\log\left(\frac{k+1}{k}\right) - \log\left(\frac{x+k-1}{k}\right)\right\} + o(1), \tag{15}$$

on account of the fact $\log(1+o(1)) = o(1)$, as $n \to \infty$.

Now put

$$u_k = (x-1)\log\left(\frac{k+1}{k}\right) - \log\left(\frac{k+x-1}{k}\right), \quad (k=1,2,..,n)$$

$$S_n = \sum_{k=1}^{n} u_k .$$

For arbitrary v_n , the following obvious identity holds.

$$S_n = \frac{1}{2} u_1 + \frac{1}{2} u_n + v_1 - v_n + \sum_{k=1}^{n-1} \{\tfrac{1}{2}u_k + \tfrac{1}{2}u_{k+1} + v_{k+1} - v_k\}.$$

Therefore, the choice

$$v_n = (x + n - 1)\log(x + n - 1) - (x + n - 1)$$

will yield

$$S_n = \tfrac{1}{2}\{(x-1)\log 2 - \log x\} + \tfrac{1}{2}\{(x-1)\log\frac{n+1}{n} - \log\frac{x+n-1}{n}\}$$

$$+ x\log x - x - (x+n-1)\log(x+n-1) + (x+n-1) + \sum_{k=1}^{n-1} w_k ,$$

where

$$w_k = (x+k-\tfrac{1}{2})\log\left(1+\frac{1}{x+k-1}\right) - 1 + \tfrac{1}{2}(x-1)\log\left(\frac{k+2}{k}\right) + \tfrac{1}{2}\log k(k+1).$$

Thus from the identities

$$\sum_{k=1}^{n-1} \log\left(\frac{k+2}{k}\right) = \log\frac{n(n+1)}{2} , \quad \sum_{k=1}^{n-1} \log k(k+1) = 2\log n! - \log n,$$

it follows that

$$S_n = \frac{1}{2}\{(x-1)\log(1+\frac{1}{n}) - \log(1+\frac{x-1}{n})\} + (x-\frac{1}{2})\log x$$

$$+ \frac{1}{2}(x-1)\log(n+1)n - (x+n-1)\log(x+n-1) + n-1 + \log n!$$

$$- \frac{1}{2}\log n + \sum_{k=1}^{n-1} \{(x+k-\frac{1}{2})\log(1+\frac{1}{x+k-1}) - 1\}.$$

On the other hand , Stirling's formula (2) which is established in the preceding section implies

$$\log n! = \frac{1}{2}\log(2\pi) - n + (n+\frac{1}{2})\log n + o(1).$$

Therefore

$$S_n = \frac{1}{2}\{(x-1)\log(1+\frac{1}{n}) - \log(1+\frac{x-1}{n})\} + \frac{1}{2}\log(2\pi) - 1$$
$$+ (x-\frac{1}{2})\log x + \{\frac{1}{2}(x-1)\log\frac{n(n+1)}{(x+n-1)^2} - n\log(1+\frac{x-1}{n})\}$$
$$+ \sum_{k=0}^{n-2} \{(x+k+\frac{1}{2})\log(1+\frac{1}{x+k}) - 1\}. \tag{16}$$

Since

$$(x-1)\log(1+\frac{1}{n}) - \log(1+\frac{x-1}{n}) = O(\frac{1}{n^2}),$$

the first curly bracket in (16) tends to 0 as $n \to \infty$.

Also the second curly bracket to $-(x-1)$ as $n \to \infty$.

On the other hand, by (12)

$$(x+k+\tfrac{1}{2})\log(1+\frac{1}{x+k}) - 1 = -\frac{1}{4(x+k)^2} + O(\frac{x+k+1/2}{(x+k)^3}).$$

Accordingly, by letting $n \to \infty$,

$$\log \Gamma(x) = \lim_{n\to\infty} S_n = \frac{1}{2}\log(2\pi) - 1 + (x-\tfrac{1}{2})\log x - (x-1)$$

$$-\frac{1}{4}\sum_{k=0}^{\infty}\frac{1}{(x+k)^2} + O(\sum_{k=0}^{\infty}\frac{x+k+1/2}{(x+k)^3})$$

$$= \frac{1}{2}\log(2\pi) + (x-\tfrac{1}{2})\log x - x + O(\tfrac{1}{x}),$$

which proves (1).

It is evident that by a slight modification a much more precise asymptotic estimate such as

$$\Gamma(x) = \sqrt{2\pi}\ x^{x-1/2}\ e^{-x+\theta/(12x)}, \quad 0<\theta<1$$

can be deducible from the above argument. Moreover, it is an important fact that this method obviously applies to complex values of x provided $\Gamma(x)$ is defined by (14).

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